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Principal Eigenvalues for Some Periodic-Parabolic Operators on $\mathbb{R}^N$ and Related Topics

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We consider a periodic-parabolic eigenvalue problem with indefinite weight function m on $\mathbb{R}^N$ ($N \geqslant 3$). Only under the assumption that the positive part of m has a certain decay we give necessary and sufficient conditions on m implying the existence of a unique positive principal eigenvalue. We prove that the zero solution of the corresponding parabolic equation looses stability at the principal eigenvalue. As a special case we obtain results on Schrödinger operators $\Delta + \lambda m$ involving a coupling parameter λ, and on the stability properties of the corresponding Schrödinger semigroup $e^{t(\Delta + \lambda m)}$. © 1995 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

We shall be concerned with a periodic-parabolic eigenvalue problem of the form

$$\begin{aligned} \partial_t \varphi(x, t) + \mathscr{A}(x, t, \partial)\, \varphi(x, t) &= \lambda m(x, t)\, \varphi(x, t) \qquad &&\text{for} \quad (x, t) \in \mathbb{R}^N \times \mathbb{R} \\ \varphi(x, t + T) &= \varphi(x, t) &&\text{for} \quad (x, t) \in \mathbb{R}^N \times \mathbb{R} \\ \lim_{|x| \to \infty} \varphi(x) &= 0 && \qquad (1.1) \end{aligned}$$

where $\mathscr{A}(x, t, \partial)$ is a differential operator of divergence form, i.e.

$$\mathscr{A}(x, t, \partial)\, u := -\sum_{i, j = 1}^{N} \partial_i (a_{ij}(x, t)\, \partial_j u). \tag{1.2}$$

We assume that the coefficients $a_{ij} = a_{ji}$ belong to $\mathrm{BUC}^{1+\alpha, 1+\alpha/2}(\mathbb{R}^N \times \mathbb{R})$ for some $\alpha \in (0, 1)$ and are T-periodic in the second variable t for some $T > 0$. By $\mathrm{BUC}^{l, l/2}$ ($l > 0$) we denote the Hölder spaces as they are introduced for example in [20], Section I.1. We also assume that $\mathscr{A}(x, t, \partial)$ is uniformly strongly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i, j = 1}^{N} a_{ij}(x, t)\, \xi^i \xi^j \geqslant \alpha\, |\xi|^2$$

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for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Further, m is a possibly indefinite weight function satisfying

$$m \in \mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R}) \tag{1.3}$$

for some $\alpha \in (0, 1)$ being T-periodic in $t \in \mathbb{R}$. By an eigenvalue of (1.1) we mean a $\lambda \in \mathbb{R}$ such that (1.1) has a nontrivial solution φ. In that case φ is called eigenfunction. If φ is nonnegative it is called a principal eigenfunction and the corresponding eigenvalue a principal eigenvalue.

The main goal of this paper is to give conditions on m involving only a decay condition on the positive part of m at infinity implying the existence of a positive principal eigenvalue λ_*. We shall prove the following theorem.

1.1. THEOREM. *Suppose that $N \geqslant 3$, $p \in [1, N/2)$ and $m = m^+ - m^-$ with m^+, m^- nonnegative,*

$$m^+ \in C_T(\mathbb{R}, L_p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)) \qquad \textit{and} \qquad m^- \in C_T(\mathbb{R}, \mathrm{BUC}(\mathbb{R}^N)). \tag{1.4}$$

Then, (1.1) has a unique positive principal eigenvalue λ_* if and only if

$$\mathscr{P}(m) := \int_0^T \sup_{x \in \mathbb{R}^n} m(x, t)\, dt > 0. \tag{1.5}$$

If $\mathscr{P}(m) \leqslant 0$ there exists no positive eigenvalue for (1.1).

Here, $C_T(\mathbb{R}, E)$ denotes the space of T-periodic continuous functions from $\mathbb{R}$ into the Banach space E. Further, $\mathrm{BUC} := \mathrm{BUC}(\mathbb{R}^N)$ is the space of bounded uniformly continuous functions on $\mathbb{R}^N$ and

$$C_0 := C_0(\mathbb{R}^N) := \{u \in \mathrm{BUC}\colon \lim_{|x| \to \infty} u(x) = 0\},$$

both equipped with the norm induced by L_∞. Finally, $L_p := L_p(\mathbb{R}^N)$ are the usual Lebesgue spaces with norm denoted by $\|\cdot\|_p$. As the results in [6] Section 3 show the restriction to dimension $N \geqslant 3$ is not for technical reasons. They show that without further conditions on the negative part of m we can not expect a similar result for $N = 1, 2$.

We shall also give a complete answer to the question of stability and instability of the zero solution of the periodic-parabolic equation

$$\begin{aligned} \partial_t u(x, t) + \mathscr{A}(x, t, \partial)\, u(x, t) &= \lambda m(x, t)\, u(x, t) \\ &\qquad \text{for} \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \\ u(x, 0) &= u_0(x) \qquad \text{for} \quad x \in \mathbb{R}^N \end{aligned} \tag{1.6}$$

if λ varies over the positive real axis. This question turns out to be closely related to the eigenvalue problem (1.1). By stability we mean stability of the solution in the L_∞-norm. We shall show that the stability of the zero solution of (1.6) changes as λ crosses the principal eigenvalue λ_*. It is well known that (1.6) has for any $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$ a unique solution ([20], Section IV.14). Let us denote the solution of (1.6) with initial time s and initial value u_0 by

$$U_\lambda(t, s)\, u_0$$

for all $t > s$. $U_\lambda(\cdot, \cdot)$ is then called the evolution operator. Concerning the stability of the zero solution for problem (1.6) we shall prove the following result.

1.2. THEOREM. *Suppose that m satisfies the hypotheses of the above theorem and that $\mathscr{P}(m) > 0$ holds. Further, denote by $\lambda_* > 0$ the unique principal eigenvalue for* (1.1). *Then for all $\lambda \in [0, \lambda_*)$ the zero solution of* (1.6) *is asymptotically stable with respect to initial conditions in $C_0(\mathbb{R}^N)$, i.e.,*

$$\lim_{t \to \infty} \| U_\lambda(t, s)\, u_0 \|_\infty = 0 \tag{1.7}$$

for all $u_0 \in C_0(\mathbb{R}^N)$. If $\lambda > \lambda_*$ the zero solution is unstable. If $\mathscr{P}(m) \leqslant 0$ then the zero solution of (1.6) *is asymptotically stable for all values of $\lambda \geqslant 0$.*

Without further assumptions on the negative part it is not possible to obtain exponential stability for the zero solution. In general we can only expect that $\mathrm{spr}(U_\lambda(T, 0)) = 1$ rather than $\mathrm{spr}(U_\lambda(T, 0)) < 1$ for $\lambda \in (0, \lambda_*)$ and hence (1.7) is not just a simple consequence of the spectral properties of the time-T-map $U_\lambda(T, 0)$.

Note that replacing λ and m by $-\lambda$ and $-m$, respectively we obtain similar results for negative λ if we impose the corresponding assumptions on $-m$.

As an important special case we obtain the existence of positive bound states for Schrödinger operators $\Delta + \lambda m$ with an indefinite potential involving a coupling parameter. Moreover, we obtain stability and instability of the zero solution of the corresponding Schrödinger semigroup

$$e^{t(\Delta + \lambda m)}. \tag{1.8}$$

The eigenvalue problem (1.1) reduces then to

$$\begin{aligned} -\Delta\varphi &= \lambda m \varphi \qquad \text{on } \mathbb{R}^N \\ \lim_{|x| \to \infty} \varphi(x) &= 0. \end{aligned} \tag{1.9}$$

As a corollary of Theorem 1.1 and 1.2 we have the following result.

1.3. THEOREM. *Let* $m \in \mathrm{BUC}(\mathbb{R}^N)$ *and* $m^+ \in C_0 \cap L_p$ *for some* $p \in [1, N/2)$. *Then,* (1.9) *has a unique principal eigenvalue* λ_* *if and only if* $m(x) > 0$ *for some* $x \in \mathbb{R}^N$. *In that case the Schrödinger semigroup* (1.8) *is stable on* C_0 *for* $\lambda \in [0, \lambda_*)$, *and unstable for* $\lambda > \lambda_*$.

Note that, by duality and interpolation, the stability properties of the Schrödinger semigroup are the same in $L_p(\mathbb{R}^N)$ for $1 < p < \infty$. In $L_1(\mathbb{R}^N)$ the semigroup in general is only bounded and not (asymptotically) stable for $\lambda \in [0, \lambda_*)$. Similar results can be obtained in the general case using the backward equation as it was done in [10]. The regularity assumptions on m can be weakened considerably. For details we refer to Remark 4.5.

The results obtained in this paper also provide the linear theory for the study of nonlinear reaction-diffusion equations on $\mathbb{R}^N$, where m is appearing from linearization at a T-periodic solution of the nonlinear equation. An application to population models with seasonal fluctuations may be found in [19].

Our results are a generalization of a similar result for the corresponding problem on a bounded domain subject to boundary conditions by Beltramo and Hess [5]. The main difficulty here is that zero lies in the spectrum of the periodic-parabolic operator $\partial_t + \mathcal{A}(x, t, \partial)$. So at a first glance the situation looks very similar to the situation in a bounded domain with Neumann boundary conditions. However, this turns out not to be true for higher dimensions. The deeper reason for that are the recurrence properties of the diffusion process making disappear heat at infinity for dimensions $N \geqslant 3$ while in the Neumann problem it is reflected at the boundary. For this reason, the problem on $\mathbb{R}^N$ behaves similarly to the Dirichlet problem in bounded domains even if there is no absorption term. Moreover, under a suitable condition on the decay of the positive part of m the same condition as in the bounded domain case turns out to be necessary and sufficient for the existence of a principal eigenvalue for (1.1). In view of the above comments a decay of m^+ is very natural to assume that the source at infinity is not too big.

Similar results for periodic-parabolic problems as ours have been obtained in [9] and [10] by other methods assuming that m^- is large enough in some sense. In [10] an exact characterization of those weights leading to exponential stability for λ between zero and the principal eigenvalue was given. Theorem 1.3 was proved in [7] for radially symmetric m having compact support, and, apart from the stability of the Schrödinger semigroup, if $|m|$ decays fast enough. Boundedness of the Schrödinger semigroup for $\lambda \geqslant 0$ smaller than the principal eigenvalue and instability for λ larger than the principal eigenvalue was obtained in [24], [25], if $m \geqslant 0$ decays fast enough (see [26], Theorem B.5.2). The main interest there was to calculate the exact divergence rate for the Schrödinger semigroup (1.8) at the principal eigenvalue. Under various conditions on m the existence of a positive principal eigenvalue of (1.9) has been shown in [1], [6], [8], [12]. All of them use Hilbert space methods which are not appropriate for periodic-parabolic problems. The existence of a unique principal eigenvalue for general "subcritical" elliptic operators on arbitrary domains was proved in [22] if m is small in some sense. To prove Theorem 1.1 we use some ideas appearing there. We point out, that in contrast to all of these authors (except [1]) we do not impose any restriction on the negative part of m apart from boundedness. As the negative part only contributes to stability a condition on its decay seems very unnatural.

1.4. *Remark*. The reason why we deal with operators of the form (1.2) is that in this case the exact global behaviour of the corresponding parabolic fundamental solution $k = k(x, t; y, s)$ is known. A result going back to [3] shows that in this case there exists a constant $c > 0$ depending only on the dimension N, the ellipticity constant and $\|a_{ij}\|_\infty$ $(i, j = 1, \ldots N)$ such that

$$c^{-1}(t-s)^{-N/2} \exp\left(-c\frac{|x-y|^2}{t-s}\right) \leqslant k(x, t; y, s) \leqslant c(t-s)^{-N/2} \exp\left(-\frac{|x-y|^2}{c(t-s)}\right), \tag{1.10}$$

(see also [11], [13], [14] and the references therein). However, our proof relies only on the upper bound. The methods apply even to more general situations, where (1.10) holds only up to translation. Situations like this occur introducing drift terms. As an example consider an elliptic operator with constant coefficients of the form

$$\mathscr{A}(\partial) = -\sum_{i,j=1}^{N} a_{ij}\partial_i\partial_j + \sum_{i=1}^{N} a_i\partial_i, \tag{1.11}$$

where $A := [a_{ij}]$ is a symmetric positive definite matrix and $a = (a_1, \ldots, a_N)$ a vector. Then, a direct calculation shows that the parabolic fundamental solution is given by

$$k(x, t; y, s) = \det A^{-1/2}(4\pi(t-s))^{-N/2} \times \exp\left(\frac{|A^{-1/2}(x-y-(t-s)\,a)|^2}{4(t-s)}\right) \tag{1.12}$$

for all $x, y \in \mathbb{R}^N$ and $s < t$. Hence it satisfies the estimate

$$c^{-1}(t-s)^{-N/2} \exp\left(-c\frac{|x-y-(t-s)\,a|^2}{t-s}\right)$$
$$\leqslant k(x, t; y, s) \leqslant c(t-s)^{-N/2} \exp\left(-\frac{|x-y-(t-s)\,a|^2}{c(t-s)}\right), \qquad (1.13)$$

where $c > 0$ is a constant depending only on the dimension N, the ellipticity constant and $\max_{i,j} |a_{ij}|$. If we replace the fixed vector a by a time dependent vector field $a(t)$ we have to replace in (1.12) and (1.13) $(t-s)a$ by the translation along the trajectory $\int_s^t a(\tau)\,d\tau$ of the vector field $a(\cdot)$. It seems to be an open question if a similar result holds in the general case of variable coefficients.

Obviously, the fundamental solution for operators of the form

$$\partial_t + \mathscr{A}(x, t, \partial) + a_0(x, t)$$

with $a_0(x, t)$ nonnegative has the same upper bound as (1.10) and (1.13), respectively. However, in general, these upper bounds are not optimal. For example, if a_0 is a positive constant, the corresponding fundamental solution has exponential decay. Necessary and sufficient conditions on a_0 leading to exponential decay were obtained in [2], [4], and [10]. In that case our method works also for $N = 1$, 2 and the decay condition on m^+ can be dropped.

In Section 2 we prove some preliminary results on the solvability of inhomogeneous periodic-parabolic equations. They are used in Section 3 to solve the eigenvalue problem (1.1) in the case m is nonnegative. In the last section we accomplish the proof of the main results. There is also some more information on the properties of the principal eigenfunction.

2. Preliminary Results

In this section we introduce some preliminary material on the solvability of the inhomogeneous T-periodic-parabolic problem

$$\begin{aligned} \partial_t u + \mathscr{A}(x, t, \partial)\, u &= f(x, t) && \text{on} \quad \mathbb{R}^N \times \mathbb{R} \\ u(x, t) &= u(x, t+T) && \text{on} \quad \mathbb{R}^N \times \mathbb{R}, \end{aligned} \qquad (2.1)$$

where $\mathscr{A}(x, t, \partial)$ is as in the previous section and f is T-periodic in t. These results shall be used in the next section to solve the eigenvalue problem (1.1) if $m \geqslant 0$. By means of the fundamental solution $k(x, t; y, s)$ associated to the parabolic operator $\partial_t + \mathscr{A}(x, t, \partial)$ the unique bounded solution of (2.1) is given by

$$u(x, t) = \int_{-\infty}^{t} \int_{\mathbb{R}^N} k(x, t; y, s) f(y, s) \, dy \, ds \tag{2.2}$$

whenever this integral exists ([20], Section V.14). It is also clear that this solution is T-periodic. The existence of a fundamental solution under our hypotheses on the coefficients of $\mathscr{A}(x, t, \partial)$ is well known ([15], [20]). Moreover,

$$k(x, t; y, s) > 0 \tag{2.3}$$

for all $x, y \in \mathbb{R}^N$ and $s < t$ ([15], Chapter 2, Theorem 11). As f is periodic in the time variable it has no decay to ensure the convergence of the integral (2.2). Also the decay of the fundamental solution given by (1.10) is not sufficient to ensure its convergence. This makes it necessary to impose assumptions on the decay of f in the space variable.

Let us consider a slightly more general problem as (2.1) by introducing a parameter dependent nonnegative zero order term a_0:

$$\begin{aligned} \partial_t u + \mathscr{A}(x, t, \partial)\, u + a_0(\mu, x, t)\, u &= f(x, t) \qquad &&\text{on} \quad \mathbb{R}^N \times \mathbb{R} \\ u(x, t) &= u(x, t+T) \qquad &&\text{on} \quad \mathbb{R}^N \times \mathbb{R}. \end{aligned} \tag{2.4}$$

The parameter μ varies in a complete metric space M and we assume throughout that

$$a_0 \in \mathrm{BUC}(M \times \mathbb{R}^N \times \mathbb{R})$$

is nonnegative and in addition satisfies a Hölder condition uniformly in $\mathbb{R}^N \times \mathbb{R}$. The fundamental solution k_μ of the operator $\mathscr{A}(x, t, \partial) + a_0(\mu)$ then satisfies

$$k_\mu(x, t; y, s) \leqslant k(x, t; y, s) \tag{2.5}$$

for all $\mu \in M$, $x, y \in \mathbb{R}^N$ and $s < t$. Moreover, as k_μ is the solution of the integral equation

$$k_\mu(x, t; y, s) = k(x, t; y, s) - \int_s^t \int_{\mathbb{R}^N} k(x, t; \xi, \tau)\, a_0(\mu, \xi, \tau)\, k_\mu(\xi, \tau; y, s) \, d\xi \, d\tau$$

it is also clear that k_μ depends continuously on $\mu \in M$. We prove now the main result of this section.

2.1. THEOREM. *Define for any $\mu \in M$ an integral operator Q_μ by*

$$Q_\mu f(x, t) := \int_s^t \int_{\mathbb{R}^N} k_\mu(x, t; y, s) f(y, s) \, dy \, ds \tag{2.6}$$

whenever the integral exists. Suppose that $N \geqslant 3$ and fix $p \in [1, N/2)$. Then, for any $\alpha \in (0, 1)$ and $q > ((1/p) - (2/N))^{-1}$ we have that

$$\begin{aligned} Q_\mu \in \mathscr{L}(&C_T(\mathbb{R}, C_0 \cap L_p), C_T(\mathbb{R}, C_0 \cap L_q)) \\ &\cap \mathscr{L}(C_T(\mathbb{R}, C_0 \cap L_p), \mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R})). \end{aligned}$$

Moreover, for any $f \in C_T(\mathbb{R}, C_0 \cap L_p)$ and $q > (1/p - 2/N)^{-1}$

$$[\mu \mapsto Q_\mu f] \in C(M, C_T(\mathbb{R}, C_0 \cap L_q)),$$

i.e. Q_μ is strongly continuous. If in addition f satisfies a Hölder condition uniformly on $\mathbb{R}^N \times \mathbb{R}$ then $u := Q_\mu f$ is a classical solution of (2.4).

Proof. Using the upper bound (1.10) (or (1.13)) of the fundamental solution and Young's inequality for convolutions we find that for any $1 \leqslant r$, $l \leqslant \infty$

$$\left\| \int_{\mathbb{R}^N} k(x, t; y, s) f(y, s) \, dy \right\|_r \leqslant c(t-s)^{-N/2(1/l - 1/r)} \|f(s)\|_l \tag{2.7}$$

for all $s < t$. The constant c depends only on r, l, N and the ellipticity constant. Applying this estimate for $r = q, \infty$ and $l = p, \infty$, respectively and (2.5) we obtain

$$\begin{aligned} \|Q_\mu f(\cdot, t)\|_r &\leqslant \int_{-\infty}^{t-1} (t-s)^{-N/2(1/p - 1/r)} \|f(s)\|_p \, ds \\ &\quad + \int_{t-1}^t (t-s)^{N/2r} \|f(s)\|_\infty \, ds \\ &\leqslant C \sup_{s \in \mathbb{R}} \max\{\|f(s)\|_\infty, \|f(s)\|_p\} \\ &= C \|f\|_{C_T(\mathbb{R}, C_0 \cap L_p)} \end{aligned}$$

for some constant $C > 0$ depending only on N, p, q and the ellipticity constant of $\mathscr{A}(x, t, \partial)$. This shows that

$$Q_\mu \in \mathscr{L}(C_T(\mathbb{R}, C_0 \cap L_p), C_T(\mathbb{R}, C_0 \cap L_q)). \tag{2.8}$$

for all $\mu \in M$ and $q > (1/p - 2/N)^{-1}$. By means of the well known identity

$$k_\mu(x, t; y, s) = \int_{\mathbb{R}^N} k_\mu(x, t; z, \tau)\, k_\mu(z, \tau; y, s)\, dz$$

we can write for any $t > 0$ and $f \in C_T(\mathbb{R}, C_0 \cap L_p)$

$$Q_\mu f(x, t) = \int_{\mathbb{R}^N} k_\mu(x, t; y, 0)\, Q_\mu f(y, 0)\, dy + \int_{\mathbb{R}^N} k_\mu(x, t; y, s)\, f(y, s)\, dy\, ds\,.$$

Using classical estimates on the fundamental solution ([20], Section IV.13), it follows that for all $\alpha \in (0, 1)$ there exists a $C > 0$ such that

$$\|Q_\mu f\|_{\mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times [T, 2T])} \leqslant C(\|Q_\mu f(\cdot, 0)\|_\infty + \|f\|_\infty)$$

By (2.8) and T-periodicity we obtain that for all $f \in C_T(\mathbb{R}, C_0 \cap L_p)$

$$\|Q_\mu f\|_{\alpha, \alpha/2} \leqslant c\, \|f\|_{C_T(\mathbb{R}, C_0 \cap L_p)}$$

for some constant $c > 0$ proving the first assertion of the theorem.

Suppose now that $f \in C_T(\mathbb{R}, C_0 \cap L_p)$ is fixed. Then the continuity of k_μ in μ shows that

$$(k_\mu(x, t; y, s) - k_\nu(x, t; y, s))\, f(y, s)$$

converges pointwise to zero as μ approaches ν. Furthermore, (2.7) and (2.5) allow us to apply Lebesgue's Theorem on Dominated Convergence to conclude that

$$\lim_{\mu \to \nu} (Q_\mu - Q_\nu) f = 0$$

in $C_T(\mathbb{R}, C_0 \cap L_q)$. This proves the strong continuity of Q_μ.

The remaining assertion on the regularity of $Q_\mu f$ follows now from classical Schauder-theory for parabolic problems ([20], Section IV.14). ∎

3. A Family of Eigenvalue Problems

In this section we study the parameter dependent eigenvalue problem

$$\begin{aligned} \partial_t \varphi + \mathscr{A}(x, t, \partial)\, \varphi + a_0(\mu)\varphi &= \lambda m \varphi && \text{on } \mathbb{R}^N \times \mathbb{R} \\ \varphi(x, t) &= \varphi(x, t + T) && \text{on } \mathbb{R}^N \times \mathbb{R}, \end{aligned} \tag{3.1}$$

where $a_0(\mu)$ is as in the previous section. For the whole section we assume that

$$m \in C_T(\mathbb{R}, C_0 \cap L_p)$$

for some $p \in [1, N/2)$. Then by Theorem 2.1 any eigenfunction φ corresponding to the eigenvalue λ satisfies the integral equation

$$\varphi = \lambda Q_\mu(m\varphi), \tag{3.2}$$

where Q_μ is the integral operator from Theorem 2.1. Obviously also the converse is true. Define now a new integral operator by setting

$$G_\mu u := Q_\mu(mu) \tag{3.3}$$

for all $u \in L_\infty$. By the definition of Q_μ it is clear that

$$|G_\mu u(x, t)| \leqslant Q_\mu(|m|)(x, t) \, \|u\|_\infty \tag{3.4}$$

holds for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $u \in C_T(\mathbb{R}, L_\infty)$. In particular, from Theorem 2.1 we find that

$$G_\mu \in \mathscr{L}(C_T(\mathbb{R}, \mathrm{BUC}), C_T(\mathbb{R}, C_0 \cap L_q))$$

whenever $q > (1/p - 2/N)^{-1}$. From (3.2) we have the following lemma.

3.1. LEMMA. *λ is a (principal) eigenvalue for (3.1) if and only if λ^{-1} is a (principal) eigenvalue of G_μ. Moreover, the corresponding eigenfunction lies in $C_T(\mathbb{R}, C_0 \cap L_q)$ for all $q > (1/p - 2/N)^{-1}$.*

This reduces the eigenvalue problem (3.1) to finding eigenvalues for the operator G_μ. The following result is the main ingredient in order to solve the eigenvalue problem.

3.2. THEOREM. *The operator $G_\mu \in \mathscr{L}(C_T(\mathbb{R}, \mathrm{BUC}), C_T(\mathbb{R}, C_0 \cap L_q))$ is compact for all $q > (1/p - 2/N)^{-1}$ and $\mu \in M$. Moreover,*

$$[\mu \mapsto G_\mu] \in C(M, \mathscr{L}(C_T(\mathbb{R}, \mathrm{BUC}), C_T(\mathbb{R}, C_0 \cap L_q)).$$

Proof. From (3.4) and Theorem 2.1 we conclude that

$$G_\mu \in \mathscr{L}(C_T(\mathbb{R}, \mathrm{BUC}), \mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R}) \cap C_T(\mathbb{R}, C_0 \times L_q)),$$

where $\alpha \in (0, 1)$ and $q > (1/p - 2/N)^{-1}$ may be chosen arbitrarily. Therefore, if (u_n) is a bounded sequence in $C_T(\mathbb{R}, \mathrm{BUC})$ then, $(G_\mu u_n)$ is a bounded sequence in $\mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R})$ as well as in $C_T(\mathbb{R}, L_q)$. It is well known that $\mathrm{BUC}^{\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R})$ is compactly embedded in the Fréchet space

$C(\mathbb{R}^{N+1})$. Hence, we may choose a subsequence of $(G_\mu u_n)$, which we denote again by $(G_\mu u_n)$ converging to a continuous function v uniformly on bounded subsets of $\mathbb{R}^{N+1}$. On the other hand, we know that $Q_\mu(|m|) \in C_T(\mathbb{R}, C_0)$. This, together with (3.4) and the T-periodicity show that $(G_\mu u_n)$ converges uniformly on $\mathbb{R}^{N+1}$ to v. In particular, we see that the family $(G_\mu u_n)$ is equicontinuous. From (3.4) we also conclude that

$$\lim_{R \to \infty} \sup_{n \in \mathbb{N}} \int_{|x| \geq R} |G_\mu u_n(x, t)|^q \, dx = 0$$

uniformly in $t \in [0, T]$. Together with the boundedness and the equicontinuity of the family this shows the relative compactness of the family $(G_\mu u_n)$ in $C_T(\mathbb{R}, L_q)$ by a well known compactness criterion. This shows the claimed compactness of G_μ.

The continuity in μ is an immediate consequence of the continuity assertion in Theorem 2.1 applied to $|m|$, and (3.4). This completes the proof of the theorem. ∎

3.3. *Remark.* We shall often consider the restrictions of the operators G_μ to

$$\mathscr{L}(C_T(\mathbb{R}, C_0)) \qquad \text{and} \qquad \mathscr{L}(C_T(\mathbb{R}, \mathrm{BUC})).$$

It is clear that the eigenvalues and the corresponding eigenfunctions are the same for all these restrictions, which means that the spectrum is independent of the choice of one of the above restrictions. We shall use this fact in the sequel without further comment.

We assume now that m is nonnegative. Then obviously G_μ is a positive operator. Unfortunately G_μ is not irreducible. So at this stage we cannot apply the powerful spectral theory for irreducible operators to ensure the existence of a unique principal eigenvalue. On the other hand if u is any nonnegative function not lying in

$$\ker G_\mu = \{u \in C_T(\mathbb{R}, C_0) : u(x, t) = 0 \text{ for all } (x, t) \in \operatorname{supp} m\} \tag{3.5}$$

then due to (2.3)

$$G_\mu u(x, t) > 0 \tag{3.6}$$

holds for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. This fact makes it possible to reduce our problem to the case of a compact irreducible operator, where spectral theory is well understood.

In the sequel we make free use of the theory of Banach lattices and positive operators. For the basic facts and definitions we refer to [23].

It is obvious that

$$E := C_T(\mathbb{R}, C_0)$$

is a Banach lattice when equipped with the natural order structure induced by the positive cone E_+ of all nonnegative functions. By (3.5) it is clear that $\ker G_\mu$ is a closed (lattice) ideal in E independent of $\mu \in M$. Therefore, the Banach space

$$\tilde{E} := E/\ker G_\mu.$$

is well defined. We define a positive cone by

$$\tilde{E}_+ = \{[u] : u \in E_+\},$$

where $[u]$ denotes the equivalence class of u. From the abstract theory we have the following result ([23], Prop. II.5.4 and Cor. 1 to Prop. II.6.4).

3.4. LEMMA. *$\tilde{E}$ is a Banach lattice. If u is a quasi-interior point of E_+ so is $[u]$ in $\tilde{E}_+$.*

Note that the quasi-interior points of the positive cone E_+ of E are those functions for which $u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

Consider now the operator $\tilde{G}_\mu$ induced by G_μ on $\tilde{E}$ which is defined by

$$\tilde{G}_\mu[u] = [G_\mu u]$$

for all $[u] \in \tilde{E}$. The next proposition reduces the eigenvalue problem (3.2) to the case of an irreducible operator on a Banach lattice.

3.5. PROPOSITION. *Let $m \in C_T(\mathbb{R}, C_0 \cap L_p)$ be nonnegative with $p \in [1, N/2)$. Then, $\tilde{G}_\mu$ is a compact irreducible operator on $\tilde{E}$. Moreover, $\sigma(\tilde{G}_\mu)\backslash\{0\} = \sigma(G_\mu)\backslash\{0\}$ and the algebraic and geometric multiplicity of the eigenvalues coincide.*

Proof. Since G_μ is compact by Theorem 3.1, the compactness of $\tilde{G}_\mu$ is obvious. From (3.6) and Lemma 3.4 it follows that $\tilde{G}_\mu$ maps every nonzero element of $\tilde{E}$ onto a quasi-interior point. In particular, this implies that $\tilde{G}_\mu$ is irreducible. The rest of the proof is standard and left to the reader. ∎

As a simple consequence we can solve our eigenvalue problem for nonnegative m.

3.6. THEOREM. *Let $m \in C_T(\mathbb{R}, C_0 \times L_p)$ for some $p \in [1, N/2)$ be nonnegative. Then, for all $\mu \in M$ (3.1) has a unique positive principal eigenvalue $\lambda_1(\mu)$ which depends continuously on $\mu \in M$.*

Moreover, the equation $(1-\lambda G_\mu)w=h$ *for* $h>0$ *has a unique positive solution if* $\lambda\in(0,\lambda_1)$, *no positive solution if* $\lambda>\lambda_1(\mu)$ *and no solution at all if* $\lambda=\lambda_1(\mu)$.

Proof. By Lemma 3.1 and Proposition 3.5 λ is a principal eigenvalue for (3.1) if and only if λ^{-1} is a principal eigenvalue for $\tilde{G}_\mu$. As $\tilde{G}_\mu$ is compact and irreducible it has positive spectral radius ([21], Theorem 4.2.2). Hence, [23], Theorem V.5.2 and its Corollary imply that $\mathrm{spr}(\tilde{G}_\mu)$ is the unique eigenvalue having positive eigenfunction and its algebraic multiplicity is one. As a consequence of the same results the assertions on the existence of positive solutions of $(1-\lambda G_\mu)w=h$ for $h>0$ follow.

The continuity of the principal eigenvalue as a function of $\mu\in M$ follows from Theorem 3.2 and [18], Section IV.3.5. ∎

3.7. Theorem. *Suppose that* M *is a subset of an Euclidean space and that* a_0 *is a concave function of* μ. *Then, under the hypotheses of the above theorem, the principal eigenvalue is a strictly concave function of* $\mu\in M$.

Proof. As concavity is defined on each segment in M we may assume without loss of generality that M is an interval in $\mathbb{R}$. Let $\mu_1, \mu_2\in M$ be arbitrary. Denote by λ_i and φ_i $(i=1,2)$ the corresponding principal eigenvalue and principal eigenfunction, respectively, and set

$$w:=\sqrt{\varphi_1\varphi_2},\qquad \bar{\lambda}:=\frac{\lambda_1+\lambda_2}{2}\qquad\text{and}\qquad \bar{\mu}:=\frac{\mu_1+\mu_2}{2}.$$

Moreover, define $v:=\bar{\lambda}G_\mu w$ and $g:=w-v$. Then, a direct calculation using the concavity of a_0 shows that

$$\partial_t g+\mathscr{A}(x,t,\partial)\,g+a_0(\bar{\mu})\,g\geqslant 0$$

on $\mathbb{R}^N\times\mathbb{R}$. We prove now that $g\geqslant 0$. Suppose this is not true and g attains a negative minimum at $(x_0,t_0)\in\mathbb{R}^N\times[0,2T]$. As g lies in $C_T(\mathbb{R},C_0)$ we can find a ball B such that $g(x,t)<g(x_0,t_0)$ for all $(x,t)\in\partial B\times[0,2T]$. Applying the strong parabolic maximum principle ([15], Chapter 2, Theorem 1) we conclude that $g(x,t)=g(x_0,t_0)$ for all $(x,t)\in B\times[0,t_0]$. But this is a contradiction to the above facts. Hence we have that

$$g=(1-\bar{\lambda}G_\mu)\,w\geqslant 0.$$

By the second part of Theorem 3.6 this is only possible for $\bar{\lambda} < \lambda_1(\bar{\mu})$. This shows the strict midpoint concavity of $\lambda_1(\cdot)$ and hence strict concavity by a well known result. ∎

The idea used in the proof of the above result is the same as in [5] and appears in a very similar form in [22].

We continue now comparing the principal eigenvalue of (3.1) with that of the Dirichlet problem on a ball $B_R := \{x \in \mathbb{R}^N : |x| < R\}$. More precisely, consider the eigenvalue problem

$$\begin{aligned} \partial_t \varphi + \mathscr{A}(x, t, \partial)\, \varphi + a_0(\mu)\, \varphi &= \lambda m \varphi && \text{on} \quad B_R \times \mathbb{R} \\ \varphi &= 0 && \text{on} \quad \partial B_R \times \mathbb{R} \\ \varphi(x, t) &= \varphi(x, t+T) && \text{on} \quad B_R \times \mathbb{R}, \end{aligned} \tag{3.7}$$

Then, it is well known ([5], [17]) that there exists for any $\mu \in M$ a unique principal eigenvalue which we denote by $\lambda_R(\mu)$. The existence of this eigenvalue could be proved by the same methods as used in this section replacing $C_0(\mathbb{R}^N)$ by $C_0(B_R)$ and the fundamental solution by the Green function $k_{\mu, R}$ for the Dirichlet problem on the ball B_R. As a consequence of the maximum principle we have that for any $R > 0$

$$k_{\mu, R}(x, t; y, s) < k_\mu(x, t; y, s) \tag{3.8}$$

for all $x, y \in \mathbb{R}^N$ and $s < t$. This fact yields the following proposition.

3.8. Proposition. *For all $R > 0$ and $\mu \in M$ we have that $\lambda_R(\mu) > \lambda_1(\mu)$, where $\lambda_1(\mu)$ denotes the principal eigenvalue for* (3.1).

Proof. Define $G_{\mu, R}$ similar as G_μ using Green's function for the Dirichlet problem on B_R in place of the fundamental solution. By (3.8) it is clear that $\|G^n_{\mu, R}\| \leqslant \|G^n_\mu\|$ for all $n \in \mathbb{N}$ and hence $\operatorname{spr}(G_{\mu, R}) \leqslant \operatorname{spr}(G_\mu)$. As $\lambda_1(\mu) = (\operatorname{spr}(G_\mu))^{-1}$ and $\lambda_R(\mu) = (\operatorname{spr}(G_{\mu, R}))^{-1}$ we conclude that $\lambda_R(\mu) \geqslant \lambda_1(\mu)$. To show the strict inequality suppose that φ and φ_R are the eigenfunctions and that $\lambda(\mu) := \lambda_1(\mu) = \lambda_R(\mu)$. Choose $\sigma > 0$ such that $v := \varphi - \sigma\varphi_R \geqslant 0$ and $\varphi - \sigma\varphi_R = 0$ at some point. As

$$\partial_t v + \mathscr{A}(x, t, \partial)\, v + a_0(\mu) = \lambda(\mu)\, v$$

on $B_R \times [0, 2T]$ and $v \geqslant 0$ on B_R the maximum principle implies that $v(x, t) > 0$ for all $(x, t) \in B_R \times (0, 2T]$ which contradicts the choice of v. Thus strict inequality holds. ∎

4. Proof of the Main Results

For the whole section we assume that $m = m^+ - m^-$, where m^+, m^- are nonnegative. Suppose also that

$$m^+ \in C_T(\mathbb{R}, L_p \cap C_0) \tag{4.1}$$

for some $p \in [1, N/2)$ and that

$$m^- \in C_T(\mathbb{R}, \mathrm{BUC}). \tag{4.2}$$

Denote by k_μ the fundamental solution for $\partial_t + \mathscr{A}(x, t, \partial) + \mu m^-$ and by G_μ the integral operator defined by (2.6). Then, Theorem 3.6 and 3.7 show that for any $\mu \geqslant 0$ there exists a unique principal eigenvalue denoted by $\lambda_1(\mu)$ for

$$\begin{aligned} \partial_t \varphi + \mathscr{A}(x, t, \partial)\, \varphi + \mu m^- \varphi &= \lambda m^+ \varphi && \text{on} \quad \mathbb{R}^N \times \mathbb{R} \\ \varphi(x, t) &= \varphi(x, t+T) && \text{on} \quad \mathbb{R}^N \times \mathbb{R}, \end{aligned} \tag{4.3}$$

which is a concave function of μ. Obviously, λ is a principal eigenvalue for (1.1) if and only if

$$\lambda_1(\lambda) = \lambda. \tag{4.4}$$

We prove now the following theorem concerning the existence of a principal eigenvalue.

4.1. Proposition. *There exists a unique principal eigenvalue $\lambda_* > 0$ for (1.1) whenever $\mathscr{P}(m) > 0$, where $\mathscr{P}(m)$ was defined in (1.5). In this case the corresponding principal eigenfunction lies in $C_T(\mathbb{R}, C_0 \cap L_q)$ for all $q > (1/p - 2/N)^{-1}$ and is unique up to scalar multiples. Moreover, $\lambda_1(\lambda) > \lambda$ for $\lambda \in [0, \lambda_*)$ and $\lambda_1(\lambda) < \lambda$ if $\lambda > \lambda_*$.*

Proof. Consider the eigenvalue problem

$$\begin{aligned} \partial_t \varphi + \mathscr{A}(x, t, \partial)\, \varphi &= \lambda m \varphi && \text{on} \quad B_R \times \mathbb{R} \\ \varphi &= 0 && \text{on} \quad \partial B_R \times \mathbb{R} \\ \varphi(x, t) &= \varphi(x, t+T) && \text{on} \quad B_R \times \mathbb{R} \end{aligned} \tag{4.5}$$

As $\mathscr{P}(m) > 0$ we can find a ball B_R such that

$$\int_0^T \max_{x \in B_R} m(x, t)\, dt > 0$$

holds. Under this condition, the main result in [5] (see [17], Theorem 16.1) shows that there exists a unique positive principal eigenvalue λ_R^* for problem (4.5). Proposition 3.8 then implies that

$$\mu := \lambda_R^* > \lambda_1(\mu).$$

As we know that $\lambda_1(0) > 0$, convexity of $\lambda_1(\cdot)$ shows that (4.4) has a unique positive solution. Thus there exists a unique positive principal eigenvalue. From the above arguments it is clear that $\lambda_1(\lambda) > \lambda$ for $\lambda \in [0, \lambda_*)$ and $\lambda_1(\lambda) < \lambda$ if $\lambda > \lambda_*$. The assertions on the eigenfunctions follow from Lemma 3.1, Theorem 3.5 and 3.6 applied for $\mu = \lambda_*$. ∎

We point out that the eigenfunction φ in general does not belong to $C_T(\mathbb{R}, L_q)$ for $q \leqslant (1/p - 2/N)^{-1}$. An example can be found in [7], Remark 2.4, where $m \in L_1$ and $\varphi \in L_q$ iff $q > (N/N-2)$.

A direct consequence of the above proposition is that the eigenfunction φ_λ to the eigenvalue $\lambda_1(\lambda)$ is a strict supersolution for (1.6) if $\lambda \in [0, \lambda_*)$ and a strict subsolution if $\lambda > \lambda_*$. This suggests some change of stability of the zero solution as λ crosses λ_*.

Note that the evolution operator

$$U_\lambda(\cdot, \cdot) \in C(\Delta, \mathscr{L}(\mathrm{BUC})) \cap C(\Delta, \mathscr{L}(C_0)),$$

where $\Delta := \{(t, s) \in \mathbb{R}^2 : s \leqslant t\}$. Moreover, the operator norm $\|U_\lambda(t, s)\|_{\infty, \infty}$ is given by

$$\|U_\lambda(t, s)\|_{\infty, \infty} = \|U_\lambda(t, s)\mathbf{1}\|_\infty$$

where $\mathbf{1}$ denotes the constant function with value 1. The norm is independent of the choice the space $\mathscr{L}(\mathrm{BUC})$ or $\mathscr{L}(C_0)$. These facts follow easily from the representation of $U_\lambda(t, s)$ by means of the fundamental solution and the properties of the fundamental solution.

We first prove our stability result.

4.2. Proposition. *For all $\lambda \in [0, \lambda_*)$ the zero solution is asymptotically stable with respect to initial values in C_0.*

Proof. In a first step we show the boundedness of the evolution operator for $\lambda \in [0, \lambda_*)$. As for this range of λ by Proposition 4.1 $\lambda < \lambda_1(\lambda)$ it follows that

$$(1 - \lambda G_\lambda)^{-1} \mathbf{1} =: \psi > 0.$$

To see this note that $\operatorname{spr} G_\lambda = \lambda_1(\lambda)^{-1} < \lambda^{-1}$, which implies that the operator $(1-\lambda G_\lambda)^{-1}$ is positive because G_λ is positive. The positivity of G_λ then also yields that

$$\psi - 1 = \lambda G_\lambda \psi > 0,$$

and in particular $\psi \geqslant 1$. Set $\psi_s := \psi(\cdot, s)$. By comparison we have that

$$\begin{aligned} U_\lambda(t,s)\,\mathbf{1} &\leqslant U_\lambda(t,s)\,\psi_s \\ &= \int_{\mathbb{R}^N} k_\lambda(\cdot, t; y, s)\,\psi_s(y)\,dy \\ &\quad + \lambda \int_s^t \int_{\mathbb{R}^N} k_\lambda(\cdot, t; y, \tau)\,m^+(y,\tau)\,\psi(y,\tau)\,dy\,d\tau \end{aligned}$$

for all $s < t$. This implies that

$$\|U_\lambda(t,s)\|_{\infty,\infty} = \|U_\lambda(t,s)\,\mathbf{1}\|_\infty \leqslant \|\psi_0\|_\infty + \lambda\,\|G_\lambda \psi\|_\infty$$

for all $s < t$ showing the boundedness of the evolution operator.

To prove asymptotic stability it is sufficient to prove (1.7) for nonnegative initial conditions u_0 in C_0. The general case follows by splitting the initial condition into positive and negative part.

As we have already seen, for $\lambda \in [0, \lambda_*)$ the principal eigenfunction for (4.3) for $\mu = \lambda$ is a strict supersolution for (1.6). Hence, $U_\lambda(T,0)\,\varphi_\lambda < \varphi_\lambda$ and the order interval $I := [0, \varphi_\lambda]$ in C_0 is invariant under $U_\lambda(T,0)$. Moreover, $U_\lambda(T,0)$ is a compact map on I. To see this observe that $U_\lambda(T,0) \in \mathscr{L}(\mathrm{BUC}, \mathrm{BUC}^\alpha)$ for any $\alpha \in (0,1)$ (c.f. [20], Section IV.14). Using a similar argument as in the proof of Theorem 3.2 compactness follows. Then, by [17], Lemma 1.1 we conclude that $U_\lambda(T,0)$ has a fixed point $\psi_0 \in I$. Defining $\psi(t) = U_\lambda(t,0)\,\psi_0$ for $t \geqslant 0$ and $\psi(t) = U_\lambda(0,t)\,\psi_0$ for $t < 0$ we easily see that ψ is a principal eigenfunction for (1.1) to the eigenvalue λ. As λ_* is the only principal eigenvalue ψ must be zero, i.e.

$$\lim_{t \to \infty} \|U_\lambda(t,s)\,\varphi_\lambda(\cdot, s)\|_\infty = 0.$$

By a simple comparison argument, (1.7) follows for any nonnegative u_0 in C_0 having compact support. Let now $u_0 \in C_0$ be arbitrary. Given $\varepsilon > 0$ we can find a decomposition $u_0 = u_1 + u_2$ such that $\|u_1\|_\infty < \varepsilon$ and u_2 having compact support. Hence, using the boundedness of the evolution operator we find that

$$\|U_\lambda(t,s)\,u_0\|_\infty \leqslant C\varepsilon + \|U(t,s)\,u_2\|_\infty$$

for some constant $C > 0$, completing the proof of the proposition. ∎

In general, asymptotic stability does not hold in BUC because $U_\lambda(T, 0)$ may have an eigenfunction in BUC to the eigenvalue 1. In fact, the existence of such eigenfunctions was used in [7] and [24] to prove boundedness of the evolution operator which in that special case is the Schrödinger semigroup. In the above proof we used a bounded supersolution instead to obtain the same result. We proceed now proving the instability of the zero solution.

4.3. PROPOSITION. *If $\lambda > \lambda_*$ the zero solution of (1.6) is unstable. More precisely, for any $\lambda > \lambda_*$ there exists a $\gamma > 0$ such that*

$$\|U_\lambda(t, s)\|_{\infty, \infty} \geqslant e^{\gamma(t-s)}$$

holds for all $s < t$.

Proof. Applying Proposition 4.1 to the operator $\partial_t + \mathscr{A}(x, t, \partial) + \gamma$ instead of $\partial_t + \mathscr{A}(x, t, \partial)$ we see that there exists a unique principal eigenvalue $\lambda^*(\gamma)$ for any $\gamma \geqslant 0$. Propositions 3.6 and 3.7 with $a_0(\mu, \gamma) := \mu m^- + \gamma$ and m replaced by m^+ imply that $\lambda^*(\gamma)$ is a continuous and concave function. In particular, it holds that

$$\lim_{\gamma \to 0} \lambda^*(\gamma) = \lambda_*.$$

On the other hand, the fact that $-\lambda m + \gamma > 0$ for γ large enough shows that

$$\lim_{\gamma \to \infty} \lambda^*(\gamma) = \infty.$$

Hence, for any $\lambda > \lambda_*$ there exists a $\gamma > 0$ such that $\lambda = \lambda^*(\gamma)$. Let φ_λ be the principal eigenfunction to $\lambda^*(\gamma)$. Then by a simple calculation

$$U_\lambda(t, s)\, \varphi_\lambda(\cdot, s) = e^{\gamma(t-s)} \varphi_\lambda(\cdot, s)$$

This completes the proof of the proposition. ∎

Note that $\operatorname{spr}(U_\lambda(T, 0)) = e^{\gamma T}$, where γ is as constructed in the proof of the above proposition. In general it is difficult to decide whether $\|U_{\lambda_*}(\cdot, \cdot)\|_{\infty, \infty}$ is bounded or not because this depends strongly on the negative part of m. Examples where it is bounded appear in [9] and [10] if the negative part is large. Examples where the evolution operator is unbounded are given in [25], Theorem 3.1 in the case $m^- = 0$, and the exact divergence rate is calculated there.

It remains to consider the case where $\mathscr{P}(m) \leqslant 0$. In this case no assumption on the decay of m^+ is necessary.

4.4. PROPOSITION. *Let $m \in C_T(\mathbb{R}, \mathrm{BUC})$ and $\mathscr{P}(m) \leqslant 0$. Then, (1.1) has no positive eigenvalue and the zero solution of (1.6) is asymptotically stable with respect to initial values in C_0 for all values of $\lambda \in [0, \infty)$.*

Proof. It is sufficient to show the asymptotic stability because this excludes the existence of eigenvalues. Moreover, splitting arbitrary initial conditions into positive and negative part we may restrict ourselves considering positive initial conditions. Setting

$$\tilde{m}(t) = \sup_{x \in \mathbb{R}^N} m(x, t)$$

we easily see that the evolution operator associated to $\partial_t + \mathscr{A}(x, t, \partial) - \lambda \tilde{m}$ is given by

$$\tilde{U}(t, s) = U_0(t, s)\, e^{\lambda \int_s^t \tilde{m}(\tau)\, d\tau}.$$

As $m \leqslant \tilde{m}$ it follows that

$$U_\lambda(nT, 0) \leqslant \tilde{U}(nT, 0) = U_0(nT, 0)\, e^{\lambda n \mathscr{P}(m)}$$

for all $n \in \mathbb{N}$. By our hypotheses we know that $e^{\lambda n \mathscr{P}(m)} \leqslant 1$ for all $n \in \mathbb{N}$ and the assertion follows from the asymptotic stability in the case $\lambda = 0$. ∎

4.5. *Remark.* If $\mathscr{A}(x, t, \partial) = \mathscr{A}(x, \partial)$ does not depend on time we could replace the parabolic fundamental solution by the elliptic and consider integral operators of the form

$$R_\mu f(x) = \int_{\mathbb{R}^N} g_\mu(x, y)\, f(y)\, dy$$

where $g(\cdot, \cdot)$ is the fundamental solution for the elliptic operator corresponding to $\mathscr{A}(x, \partial) + a_0(\mu, x)$. A similar procedure as in Sections 2–4 would lead to the same results in the autonomous case. In fact, both methods are equivalent, the one using the semigroup, and the other the resolvent of its generator.

Further, an inspection of the proofs also shows that $m \in L_\infty$ and $m^+ \in L_p \cap L_q$ for some $1 \leqslant p < N/2 < q < \infty$ is sufficient. Condition (1.5) has then to be replaced by the assumption that the measure of $\{x \in \mathbb{R}^N : m(x) > 0\}$ is positive. The comparison with the Dirichlet problem on a sufficiently large ball in the proof of Proposition 4.1 still works using [16], Theorem 2 instead of [5]. In fact, in [25] it is assumed that $m \in L_p \cap L_q$ for p, q as above, in particular also the negative part of m.

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